Matrices - Part A

1. Find the eigenvalues of the matrix $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$. [N/D-2014]

Ans: Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

$S_1 = 1+5+1 = 7$; $S_2 = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = (5-1)+(1-9)+(5-1) = 0$

$S_3 = 1(5-1) - 1(1-3) + 3(1-15) = 4+2-42 = -36$

The characteristic eqn. is $\lambda^3 - 7\lambda^2 + 36 = 0$

$$\begin{array}{c|cccc} \lambda = 3 & 1 & -7 & 0 & 36 \\ & & 3 & -12 & -36 \\ \hline & 1 & -4 & -12 & \boxed{0} \end{array}$$

$$\begin{array}{c|c} \times & + \\ -12 & -4 \\ \hline -6 & +2 \\ \lambda-6 & \lambda+2 \end{array}$$

$\lambda^2 - 4\lambda - 12 = 0$

$(\lambda-6)(\lambda+2) = 0$ $\therefore \lambda = -2, 3, 6$

Hence the eigenvalues are $-2, 3$ & 6.

2. State Cayley-Hamilton theorem. [N/D-2014] [A/M-2015] [M/J-2010]

Ans: Every square matrix satisfies its own characteristic equation.

3. Use C-H thm., to find $A^4 - 8A^3 - 12A^2$ when $A = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}$. [N/D-2010]

Ans: Given $A = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}$

$S_1 = 5+3 = 8$; $S_2 = |A| = 15-3 = 12$

The characteristic eqn. is $\lambda^2 - 8\lambda + 12 = 0$. Use C-H thm., we get $A^2 - 8A + 12I = 0$ —(1)

$$\begin{array}{r|l} & A^2 \\ \hline A^2-8A+12I & A^4-8A^3-12A^2 \\ & A^4-8A^3+12A^2 \\ & (-)\;(+)\;\;(-) \\ \hline & -24A^2 \end{array}$$

$\therefore A^4 - 8A^3 - 12A^2 = (A^2 - 8A + 12)A^2 - 24A^2$

$= 0 - 24A^2$ ($\because$ by (1))

$= -24A^2$

$\therefore A^4 - 8A^3 - 12A^2 = -24A^2 = -24\begin{pmatrix} 28 & 24 \\ 8 & 12 \end{pmatrix} = \begin{pmatrix} -672 & -576 \\ -192 & -288 \end{pmatrix}$

4. Use C-H thm., to find $A^4 - 4A^3 - 5A^2 + A + 2I$ when $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. [M/J-2009]

Ans: Given $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

$S_1 = 1+3 = 4$; $S_2 = |A| = 3-8 = -5$. The characteristic eqn. is $\lambda^2 - 4\lambda - 5 = 0$

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Use C-H thm. we get $A^2-4A-5I=0$ —(1)

$$\begin{array}{r|l} & A^2 \\ \hline A^2-4A-5I & A^4-4A^3-5A^2+A+2I \\ & A^4-4A^3-5A^2 \\ & (-)\ (+)\quad (+) \\ \hline & A+2I \\ \hline \end{array}$$

$\therefore A^4-4A^3-5A^2+A+2I = (A^2-4A-5I)A^2+A+2I = 0+A+2I$ ($\because$ by (1))

$$= \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$$

(5) Check whether the matrix B is orthogonal? Justify. $B=\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. [Jan-2009]

Ans: To check: $BB^T = B^TB = I$

$$BB^T = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta+\sin^2\theta & -\sin\theta\cos\theta+\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta+\sin\theta\cos\theta & \sin^2\theta+\cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Similarly we get $B^TB = I$. Hence the given matrix B is orthogonal.

(6) If 3 & 5 are 2 eigenvalues of the matrix $A=\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ then find its 3rd eigenvalue & hence $|A|$. [A/M 2018] [A/M-2017]

Ans: Given $\lambda_1=3$, $\lambda_2=5$

Sum of the eigenvalues = Sum of the main diagonal elements

$$\lambda_1+\lambda_2+\lambda_3 = 8+7+3 = 18$$

$$3+5+\lambda_3 = 18 \Rightarrow \lambda_3 = 18-8 = 10$$

$|A|$ = Product of eigenvalues $= 3\times5\times10 = 150$

(7) Write down the matrix corresponding to the quadratic form $2x_1^2+5x_2^2+4x_1x_2+2x_3x_1$. [N/D-2016]

Ans: $A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

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8) If 2, 3 are the eigenvalues of $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ b & 0 & 2 \end{pmatrix}$, then find the value of b. [N/D-2014]

Ans: Given $\lambda_1 = 2$, $\lambda_2 = 3$

Sum of the eigenvalues = Sum of the main diagonal elements $= 2+2+2 = 6$

$5 + \lambda_3 = 6 \Rightarrow \lambda_3 = 6-5 = 1$

Product of eigenvalues $= |A| = 2(4-0) + 1(0-2b) = 8-2b$

$2 \times 3 \times 1 = 8-2b \Rightarrow 6 = 8-2b \Rightarrow 2b = 8-6 = 2 \Rightarrow \boxed{b=1}$

9) If the eigenvalues of the matrix A of order 3×3 are 2, 3 & 1, then find the eigenvalues of adjoint of A. [M/J-2016]

Ans: WKT $A^{-1} = \frac{1}{|A|} \operatorname{Adj} A \Rightarrow \operatorname{Adj} A = |A| A^{-1}$

$|A|$ = Product of the eigenvalues $= 2 \times 3 \times 1 = 6$

Eigenvalues of A^{-1} are $\frac{1}{2}, \frac{1}{3}$ & 1.

$\therefore$ Eigenvalues of adjoint of A are $\frac{6}{2}, \frac{6}{3}$ & 6 (ie) 3, 2 & 6.

10) If λ is the eigenvalue of the matrix A, then prove that λ^2 is the eigenvalue of A^2. [M/J-2016] [M/J-2014]

Ans: Given λ is the eigenvalue of A. Let the corresponding eigenvector be X.

Then we have $AX = \lambda X$ —(1)

Pre-multiplying by A on both sides, we get

$AAX = A\lambda X \Rightarrow A^2X = \lambda AX = \lambda(\lambda X)$ ($\because$ by (1)) $= \lambda^2 X$

$\therefore A^2 X = \lambda^2 X$

From this we get, λ^2 is the eigenvalue of A^2.

11) Find the eigenvalues of the inverse of the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix}$ [M/J-2014]

Ans: Given matrix A is a upper triangular matrix.

$\therefore$ Eigenvalues of A = Entries in main diagonal.

$\therefore$ Eigenvalues of A are 2, 3 & 4.

$\therefore$ Eigenvalues of A^{-1} are $\frac{1}{2}, \frac{1}{3}$ & $\frac{1}{4}$.

Hence the eigenvalues

12) If 2, −1, −3 are the eigenvalues of the matrix A, then find the eigenvalue of the matrix $A^2 - 2I$. [M/J-2014]

Ans: Eigenvalues of A^2 are $2^2, (-1)^2$ & $(-3)^2$ (ie) 4, 1 & 9

Eigenvalues of $2I$ are 2, 2 & 2. ($\because$ Eigenvalues of I are 1, 1 & 1)

Eigenvalues

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Hence the eigenvalues of A^2-2I are $4-2, 1-2$ & $9-2$ (ie) $2, -1$ & 7.

(13) Find the eigenvalues of $3A+2I$, where $A=\begin{pmatrix}5 & 4\\ 0 & 2\end{pmatrix}$. [N/D-2015]

Ans: Given the eigenvalues of A are 5 & 2. ($\because$ A is a upper triangular matrix)

Eigenvalues of 3A are 15 & 6.

Eigenvalues of 2I are 2 & 2.

Hence the eigenvalues of $3A+2I$ are 17 & 8.

(14) What is the nature of the quadratic form $x^2+y^2+z^2$ in 4 variables? [N/D-2015]

Ans: The quadratic form $x^2+y^2+z^2$ in 4 variables $= x^2+y^2+z^2+0t^2$

= canonical form

Canonical form contains 3 +ve terms & one zero.

$\therefore$ Quadratic form is said to be positive semi-definite.

(15) Find the constants a & b such that the matrix $\begin{pmatrix}a & 4\\ 1 & b\end{pmatrix}$ has 3 & -2 as its eigenvalues. [A/M-2017]

Ans: Sum of the eigenvalues = Sum of the main diagonal elements

$3-2 = a+b \Rightarrow a+b=1$ —(1)

Product of the eigenvalues $= |A| = ab-4 = 3\times -2 \Rightarrow ab-4=-6 \Rightarrow ab=-2$ —(2)

From (2), $a = \frac{-2}{b}$

$\therefore$ (1) becomes, $\frac{-2}{b}+b=1 \Rightarrow -2+b^2=b \Rightarrow b^2-b-2=0$

$(b+1)(b-2)=0$

$b=-1, 2$

$\therefore a = 1-b \Rightarrow a = 2, -1$

$\times$ $+$
-2 | -1
$\frac{+1}{b+1}$ | $\frac{-2}{b-2}$

Hence $a=2, b=-1$ (or) $a=-1, b=2$

(16) Find the sum & product of the eigenvalues of $A=\begin{pmatrix}2 & 1 & 2\\ 1 & 3 & 1\\ 2 & 1 & -6\end{pmatrix}$. [M/J-2014]

Ans: Sum of the eigenvalues = Sum of the main diagonal elements $=2+3-6=-1$

Product of the eigenvalues $= |A| = 2(-18-1)-1(-6-2)+2(1-6)$

$= 2(-19)-1(-8)+2(-5) = -40$

(17) Give the nature of a quadratic form whose matrix is $\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -2\end{pmatrix}$. [A/M-2015]

Ans: Given matrix is a diagonal matrix.

$\therefore$ Eigenvalues of the matrix are $-1, -1$ & -2.

Canonical form is $-y_1^2-y_2^2-2y_3^2$. Canonical form contains 3 -ve terms.

$\therefore$ Quadratic form is said to be negative definite.

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18) The product of 2 eigenvalues of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16. Find the 3^{rd} eigenvalue. [N/D-2015]

Ans: Given $\lambda_1 \lambda_2 = 16$ —(1)

Product of eigenvalues $= |A| = 6(9-1)+2(-6+2)+2(2-6) = 48-8-8 = 32$

$\lambda_1 \lambda_2 \lambda_3 = 32$.

$16\lambda_3 = 32 \; (\because \text{by } (1)) \Rightarrow \lambda_3 = \frac{32}{16} = 2 \quad \therefore \lambda_3 = 2$

19) If λ be an eigenvalue of a non-singular matrix A, show that λ^{-1} is an eigenvalue of A^{-1}. [N/D-2014]

Ans: Given λ is an eigenvalue of A. Let the corresponding eigenvector be X.

Then we have $AX = \lambda X$.

Pre-multiplying by A^{-1} on both sides, we get

$A^{-1}AX = \lambda A^{-1}X \Rightarrow IX = \lambda A^{-1}X \Rightarrow X = \lambda A^{-1}X$

Multiplying by $\frac{1}{\lambda}$ on both sides, $\frac{1}{\lambda}X = A^{-1}X$

From this we get, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

20) Write down the quadratic form corresponding to the matrix $A = \begin{pmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{pmatrix}$. [M/J-2012]

Ans: $Q.F = 0x_1^2 + x_2^2 + 2x_3^2 + (2\times5)x_1x_2 + (-1\times2)x_1x_3 + (2\times6)x_2x_3$

$\therefore Q.F = x_2^2 + 2x_3^2 + 10x_1x_2 - 2x_1x_3 + 12x_2x_3$

21) Show that the eigenvalues of a null matrix are zero. [A/M 2018]

Ans: Let λ be an eigenvalue of A & the corresponding eigenvector be X.

Then we have $AX = \lambda X$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (\because A \text{ is a null matrix})$$

$0x_1 + 0x_2 = \lambda x_1$

$0x_1 + 0x_2 = \lambda x_2$

$\because X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq 0 \Rightarrow \lambda = 0$. Hence the eigenvalues of a null matrix are zero.

22) If the sum of 2 eigenvalues & trace of a matrix A are equal, find the value of $|A|$. [N/D-2016] [M/J-2009]

Ans: Given $\lambda_1 + \lambda_2$ = trace of a matrix A = Sum of the main diagonal elements

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WKT $\lambda_1+\lambda_2+\lambda_3$ = Sum of the main diagonal elements.

From this we get, $\lambda_3 = 0$

$|A|$ = Product of the eigenvalues $= 0$ $(\because \lambda_3 = 0)$

(23) When is a Quadratic form said to be singular? What is its rank then? [N/D-2015]

Ans: A Quadratic form is said to be singular if atleast one of the eigenvalues zero. $\therefore$ Rank is $< n$.

(24) For what values of 'c' the eigenvalues of the matrix $\begin{pmatrix} 1 & 2 \\ c & 4 \end{pmatrix}$ are real & unequal, real & equal, complex conjugates? [Jan-2011]

Ans: Let $A = \begin{pmatrix} 1 & 2 \\ c & 4 \end{pmatrix}$

$S_1 = 5$, $S_2 = 4-2c$. Hence the characteristic eqn. is $\lambda^2 - 5\lambda + 4 - 2c = 0$

$$\lambda = \frac{5 \pm \sqrt{25-4(4-2c)}}{2} = \frac{5 \pm \sqrt{25-16+8c}}{2} = \frac{5 \pm \sqrt{9+8c}}{2}$$

Eigenvalues are $\frac{5+\sqrt{9+8c}}{2}$ & $\frac{5-\sqrt{9+8c}}{2}$

Real & equal: $\frac{5+\sqrt{9+8c}}{2} = \frac{5-\sqrt{9+8c}}{2} \Rightarrow \sqrt{9+8c} = -\sqrt{9+8c} \Rightarrow 9+8c=0 \Rightarrow c = -9/8$

Complex conjugate: $9+8c<0 \Rightarrow 8c<-9 \Rightarrow c<-9/8$

Real & unequal: $9+8c>0 \Rightarrow 8c>-9 \Rightarrow c>-9/8$

(25) If A is an nxn real symmetric matrix, D is an nxn diagonal matrix whose diagonal elements are the eigenvalues of the matrix A & P is an nxn orthogonal diagonalizing matrix whose columns are the normalized eigenvectors of the matrix A, satisfying the similarity transformation $D = P^{-1}AP$, then find the matrix A^k, where k is a +ve integer? [Jan-2011]

Ans: $A^k = PD^kP^{-1}$

(26) Can $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be diagonalized? why? [Jan-2012] [Jan-2010]

Ans: Given matrix A can be diagonalisable, because it is a diagonal matrix.

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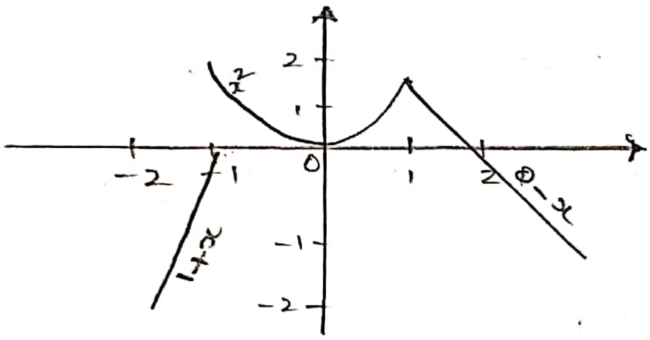
UNIT - II

DIFFRENTIAL CALCULUS

1. Sketch the graph of the function $f(x) = \begin{cases} 1+x & ; x < -1 \\ x^2 & ; -1 \leq x \leq 1 \\ 2-x & ; x \geq 1 \end{cases}$

and use it to determine the value of 'a' for which $\lim_{x \to a} f(x)$ exists? [Jan - 2018]

Soln:-



From the graph $\lim_{x \to a} f(x)$ exists $\forall$ 'a' except when $a = -1$.

($\because$ the left & right limits are different at $a = -1$)

2. Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so where? [Jan - 2018]

Soln:- The horizontal tangent occurs when $\frac{dy}{dx}$ is zero.

$$\frac{dy}{dx} = 4x^3 - 4x = 0 \Rightarrow x = 0 \text{ (or) } x = \pm 1.$$

$\therefore$ The given curve has horizontal tangent at $x = 0, 1$ & -1.

The corresponding points on curve are $(0,2), (1,1), (-1,1)$

3. Evaluate $\lim_{x \to a} \left(\frac{x^{5/8} - a^{5/8}}{x - a} \right)$

Soln:

$$\lim_{x \to a} \left(\frac{x^{5/8} - a^{5/8}}{x - a} \right) = \frac{a^{5/8} - a^{5/8}}{a - a} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ By L-Hospital's rule

$$\lim_{x \to a} \left(\frac{x^{5/8} - a^{5/8}}{x - a} \right) = \lim_{x \to a} \left(\frac{\frac{d}{dx}\left(x^{5/8} - a^{5/8}\right)}{\frac{d}{dx}(x - a)} \right) = \lim_{x \to a} \left(\frac{\frac{5}{8} x^{\frac{5}{8} - 1}}{1} \right)$$

$$= \frac{5}{8} a^{\frac{5}{8} - 1}$$

$$= \frac{5}{8 a^{3/8}}$$

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4. Find the slope of the tangent line to the parabola $y = 4x - x^2$ at $(1,3)$

Soln

m = slope of tangent = $\frac{dy}{dx}$ at given point.

$$\therefore \quad m = \left(\frac{dy}{dx}\right)_{(1,3)} = (4 - 2x)_{(1,3)} = 4 - 2 = 2.$$

5. Find the 50^{th} derivative of $y = \cos 2x$.

Soln:-

$y = \cos 2x$, $y' = -2\sin 2x$, $y'' = -(2)^2 \cos 2x$, $y''' = (2)^3 \sin 2x$

$y^{IV} = (2)^4 \cos 2x$

Hence $y^{(48)} = (2)^{48} \cos 2x$

$y^{(49)} = -(2)^{49} \sin 2x$

$y^{(50)} = -(2)^{50} \cos 2x$.

6. Find the critical numbers of $f(x) = x^{3/4} - 2x^{1/4}$.

Soln

$$f'(x) = \frac{3}{4} x^{3/4 - 1} - 2 \times \frac{1}{4} x^{1/4 - 1}$$

$$f'(x) = \frac{3}{4x^{1/4}} - \frac{1}{2x^{3/4}}$$

$$f'(x) = 0 \implies \frac{3}{4x^{1/4}} = \frac{1}{2x^{3/4}}$$

$$\implies 3 \times 2 x^{3/4} = 4x^{1/4}$$

$$\implies x^{1/2} = 2/3$$

$$\implies x = 4/9.$$

Also $f'(x)$ does not exist when $x = 0$.

$\therefore$ Critical numbers are $0, \frac{4}{9}$.

7. Rolle's Thm:- (Statement)

If (i) $f(x)$ is continuous on $[a,b]$

(ii) $f(x)$ is differentiable on (a,b)

and (iii) $f(a) = f(b)$, then there is a point $c \in (a,b)$ such that $f'(c) = 0$

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8. Lagrange Mean Value Theorem :- (Statement)

If (i) $f(x)$ is continuous on $[a,b]$

(ii) $f(x)$ is differentiable on (a,b), then there is a point $c \in (a,b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

9. Squeeze Thm (or) Sandwich Theorem :- (or) Pinching Theorem :-

If $f(x) \le g(x) \le h(x)$ when x is near 'a' and $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = L$.

10. Evaluate $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$

Soln :-

Since $\sin\left(\frac{1}{x}\right)$ does not tend to any limit as $x\to 0$, the normal method fails.

W.K.T $\quad -1 \le \sin\left(\frac{1}{x}\right) \le 1$.

Multiplying by x^2, $\quad -x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$

Since $\lim_{x\to 0} (-x^2) = 0 = \lim_{x\to 0} (x^2)$, by squeeze thm

$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

11) Determine whether each of the following functions is even, odd or neither even nor odd. (a) $f(x) = x\cos x$ (b) e^{x^2} (c) $f(x) = 2x+1$

Soln :-

(a) $f(x) = x\cos x \Rightarrow f(-x) = -x.\cos(-x) = -x.\cos x = -f(x)$

$\therefore f(x)$ is an odd function.

(b) $f(x) = e^{x^2} \Rightarrow f(-x) = e^{(-x)^2} = e^{x^2} = f(x)$

$\therefore f(x)$ is an even function.

(c) $f(x) = 2x+1 \Rightarrow f(-x) = -2x+1 \neq -f(x)$

Also $f(-x) \neq f(x)$

$\therefore f(x)$ is neither even nor odd.

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12) Check whether $\lim_{x \to 0} \frac{|x|}{x}$ exist (or) not (4)

Soln:-

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1 \quad , \quad \lim_{x \to 0^-} \frac{|x|}{x} = -1.$$

Since right limit $\neq$ left limit, $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

13) Evaluate $\lim_{x \to \pi/2} \frac{1+\cos 2x}{(\pi - 2x)^2}$

Soln:-

$$\lim_{x \to \frac{\pi}{2}} \left[\frac{1+\cos 2x}{(\pi-2x)^2}\right] = \left[\frac{1+\cos 2(\pi/2)}{(\pi - 2(\pi/2))^2}\right] = \left[\frac{1+\cos\pi}{(\pi-\pi)^2}\right] = \frac{0}{0} \text{ (indet. form)}$$

$$\therefore \lim_{x \to \pi/2} \left[\frac{1+\cos 2x}{(\pi-2x)^2}\right] = \lim_{x \to \pi/2} \left[\frac{\frac{d}{dx}(1+\cos 2x)}{\frac{d}{dx}(\pi-2x)^2}\right] = \lim_{x \to \pi/2}\left[\frac{-2\sin 2x}{2(\pi-2x).(-2)}\right] = \frac{0}{0} \text{ (ind. form)}$$

$$\therefore \lim_{x \to \pi/2} \left[\frac{1+\cos 2x}{(\pi-2x)^2}\right] = \lim_{x \to \pi/2} \left[\frac{(d^2/dx^2)(1+\cos 2x)}{d^2/dx^2(\pi-2x)}\right] = \lim_{x \to \pi/2}\left(\frac{-4\cos 2x}{-4(-2)}\right) = \frac{-4.\cos\pi}{8} = \frac{-4(-1)}{8} = \frac{1}{2}$$

$$\therefore \lim_{x \to \pi/2} \left[\frac{1+\cos 2x}{(\pi-2x)^2}\right] = \frac{1}{2}.$$

14) ~~Indet~~ Intermediate Value Theorem:-

If $f(x)$ is a continuous function on $[a,b]$ and N is any number lies between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$ then there is a point $c \in (a,b)$ with $f(c) = N$.

15) Show that there is a root of the eqn $x^3 - x - 1 = 0$ between 1 and 2.

Soln:-

Take $f(x) = x^3 - x - 1$

$f(1) = 1 - 1 - 1 = -1$, $f(2) = 8 - 2 - 1 = 5$.

$\Rightarrow f(1) < f(2)$ also 0 lies between $f(1)$ and $f(2)$

$\therefore$ There is a number c between 1 and 2 with $f(c) = 0$.

($\because$ using Intermediate Value thm.)

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(16) Given that $\lim_{x \to 2} f(x) = 4$ and $\lim_{x \to 2} g(x) = -2$. Find the limit $\lim_{x \to 2} \left[\frac{3f(x)}{g(x)}\right]$

— [Nov/Dec - 2019]

Soln

Given $\lim_{x \to 2} f(x) = 4$ and $\lim_{x \to 2} g(x) = -2$.

$$\therefore \lim_{x \to 2} \left[\frac{3f(x)}{g(x)}\right] = \frac{\lim_{x \to 2}(3f(x))}{\lim_{x \to 2} g(x)} = \frac{3.\lim_{x \to 2} f(x)}{\lim_{x \to 2} g(x)} = \frac{3(4)}{-2} = -6$$

(17) If $f(x) = xe^x$ then find the expression for $f''(x)$. [Nov/Dec - 2019]

Soln:

Given $f(x) = xe^x$

$$\Rightarrow f'(x) = xe^x + e^x(1) = xe^x + e^x$$

$$\therefore f'(x) = e^x(x+1) \longrightarrow (1)$$

Differentiating (1) w.r.t 'x'

$$f''(x) = e^x(1) + (x+1)e^x = e^x(1+x+1) = e^x(x+2)$$

(18) Check whether $\lim_{x \to -3} \left[\frac{3x+9}{|x+3|}\right]$ exist. [A/M - 2019]

Soln:

$$\lim_{x \to -3^-} \left[\frac{3x+9}{|x+3|}\right] = \lim_{x \to -3^-} \left[\frac{3x+9}{-(x+3)}\right] = \lim_{x \to -3^-} \left[\frac{3(x+3)}{-(x+3)}\right] = -3$$

$$\lim_{x \to -3^+} \left[\frac{3x+9}{|x+3|}\right] = \lim_{x \to -3^+} \left[\frac{3x+9}{x+3}\right] = \lim_{x \to -3^+} \left[\frac{3(x+3)}{x+3}\right] = 3.$$

Since $\lim_{x \to -3^-} \left(\frac{3x+9}{|x+3|}\right) \neq \lim_{x \to -3^+} \left(\frac{3x+9}{|x+3|}\right)$, limit doesnot exist.

(19) Find the critical points of $y = 5x^3 - 6x$. [A/M - 2019]

Soln:

Given $y = 5x^3 - 6x$.

Critical points : $y' = 0$ (or) $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = 15x^2 - 6$$

$$\frac{dy}{dx} = 0 \Rightarrow 15x^2 - 6 = 0$$

$$\Rightarrow 5x^2 - 2 = 0$$

$$\Rightarrow x^2 = 2/5 \Rightarrow x = \pm\sqrt{2/5}$$ are critical points.

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(20) Find the domain of $f(x) = \sqrt{3-x} - \sqrt{2+x}$ [Nov/Dec - 2018]

Soln

Given $f(x) = \sqrt{3-x} - \sqrt{2+x}$

Here $3-x \geq 0$ and $2+x \geq 0$.

$\Rightarrow 3 \geq x$ and $x \geq -2$

$\Rightarrow -2 \leq x \leq 3$.

$\therefore$ Domain is $[-2, 3]$

---

(21) Evaluate $\lim_{t\to 1} \frac{t^4-1}{t^3-1}$. [Nov/Dec - 2018]

Soln

$$\lim_{t\to 1} \frac{t^4-1}{t^3-1} = \frac{0}{0} \text{ (indeterminate form)}$$

$$\therefore \lim_{t\to 1} \left(\frac{t^4-1}{t^3-1}\right) = \lim_{t\to 1} \left(\frac{4t^3}{3t^2}\right) = 4/3$$

Hint

Suppose $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right]$ is indeterminate form

Then $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

---

(22) If $xe^y = x-y$ then find $\frac{dy}{dx}$ by implicit differentiation. [N/D-2020]

Soln

Given $xe^y = x-y \longrightarrow (1)$

Diff (1) w.r.t 'x'

$$xe^y \frac{dy}{dx} + e^y(1) = 1 - \frac{dy}{dx}.$$

$$\Rightarrow xe^y \frac{dy}{dx} + \frac{dy}{dx} = 1 - e^y.$$

$$\Rightarrow (xe^y+1)\frac{dy}{dx} = 1-e^y.$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-e^y}{xe^y+1}$$

---

(23) If $\lim_{x\to 1} \left[\frac{f(x)-8}{x-1}\right] = 10$ then find $\lim_{x\to 1} f(x)$. [N/D-2020]

Soln:

Given $\lim_{x\to 1} \left[\frac{f(x)-8}{x-1}\right] = 10$.

$$\Rightarrow \frac{\lim_{x\to 1} [f(x)-8]}{\lim_{x\to 1} (x-1)} = 10$$

$$\Rightarrow \lim_{x\to 1} (f(x)-8) = 10 \lim_{x\to 1} (x-1) = 0.$$

$$\Rightarrow \lim_{x\to 1} f(x) = 8$$

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UNIT - III

FUNTION OF SEVERAL VARIABLES

1) If $x = r^2 - \theta^2$, $y = 2r\theta$ find $\frac{\partial(r,\theta)}{\partial(x,y)}$ [April / May - 2018] [Jan - 2018]

Soln:-

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\partial(x,y)/\partial(r,\theta)}$$

Now $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2r & -2\theta \\ 2\theta & 2r \end{vmatrix} = 4r^2 + 4\theta^2$$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{4r^2+4\theta^2}$$

2. When is a function said to be stationary at a point (x,y)? [April/May - 2018] [Jan - 2018]

Soln:- A function 'f' is stationary at a point (x,y) if $\frac{\partial f}{\partial x}$ at the point $(x,y) = 0$ & $\frac{\partial f}{\partial y}$ at the point $(x,y) = 0$

3. Find $\frac{du}{dt}$ when $u = x^2 y$, $x = t^2$ and $y = e^t$. [NOV/DEC - 2017]

Soln

$u = x^2 y$, $x = t^2$, $y = e^t \Rightarrow u = t^4 e^t$

$$\therefore \frac{du}{dt} = t^4 e^t + 4t^3 e^t$$

4. If $x = u(1+v)$ and $y = v(1+u)$ find $\partial(x,y)/\partial(u,v)$ [NOV/DEC - 2017]

Soln:- Given $x = u + uv$, $y = v + uv$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = (1+u)(1+v) - uv$$

$$= 1 + u + v + uv - uv$$

$$= 1 + u + v.$$

5. If $u = \sin^{-1}\left(\frac{x^3 - y^3}{x+y}\right)$ then prove that $x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u$. [April / May - 2017]

Soln:-

Let $f(x,y) = \sin u = \left(\frac{x^3 - y^3}{x+y}\right) \longrightarrow (1)$

$$f(tx, ty) = \frac{t^3x^3 - t^3y^3}{tx + ty} = t^2\left(\frac{x^3 - y^3}{x+y}\right)$$

$\therefore f$ is a homogeneous function of degree 2 in (x, y).

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$\therefore$ By Euler Thm $x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = 2f \rightarrow (2)$

$$\frac{\partial f}{\partial x} = \cos u.\frac{\partial u}{\partial x} \; ; \quad \frac{\partial f}{\partial y} = \cos u.\frac{\partial u}{\partial y} \rightarrow (3)$$

Sub (3) in (2) $\quad x\cos u.\frac{\partial u}{\partial x} + y.\cos u.\frac{\partial u}{\partial y} = 2\sin u$

$$\Rightarrow \quad x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u.$$

H/p.

6. Find $\frac{du}{dt}$ if $u = x/y$ where $x = e^t$, $y = \log t$. [Apr/May – 2017]

Soln

$u = x/y$, $x = e^t$, $y = \log t$

$$\Rightarrow \quad u = e^t/\log t \quad \Rightarrow \quad \frac{du}{dt} = \frac{\log t\,(e^t) - e^t\,(1/t)}{(\log t)^2}.$$

7. Find $\frac{du}{dt}$ when $u = x^2+y^2$, $x = at^2$, $y = 2at$ [Nov/Dec – 2016] [Apr/May – 2015]

Soln

$u = x^2+y^2$, $x = at^2$, $y = 2at \quad \Rightarrow \quad u = a^2t^4 + 4a^2t^2$

$$\Rightarrow \frac{du}{dt} = 4a^2t^3 + 8a^2t.$$

8. If $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(x,y)}{\partial(r,\theta)}$ [Nov/Dec – 2016] [Dec – 2015] [Nov/Dec – 2014] [Jan – 2014]

Soln:-

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

9. If $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(r,\theta)}{\partial(x,y)}$. [May / June – 2016]

Soln

W.K.T $\frac{\partial(x,y)}{\partial(r,\theta)} = r$ (Refer prob: 8)

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = 1/r.$$

10. If $x^2+y^2 = 1$ then find $\frac{dy}{dx}$. [May/June – 2016]

Soln:-

Diff $x^2+y^2=1$ w.r.t 'x' we get, $2x + 2y\frac{dy}{dx} = 0.$

$$\Rightarrow x + y\frac{dy}{dx} = 0$$

$$\Rightarrow dy/dx = -x/y.$$

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11. If $x^y + y^x = 0$, then find $\frac{dy}{dx}$. [Dec - 2015]

Soln

Given $x^y = -y^x$. Taking logarithm on both sides we get

$$\log x^y = \log(-y^x)$$

$$\Rightarrow y \cdot \log x = x \log(-y) \longrightarrow ①$$

Diff ① w.r.t 'x'

$$y\left(\frac{1}{x}\right) + \log x \cdot \frac{dy}{dx} = x\left(\frac{1}{-y}\right)\frac{dy}{dx} + \log(-y) \ (1)$$

$$\log x \cdot \frac{dy}{dx} + \frac{x}{y}\frac{dy}{dx} = \log(-y) - y/x$$

$$\frac{dy}{dx}\left[\log x + x/y\right] = \log(-y) - y/x.$$

$$\Rightarrow dy/dx = \frac{\log(-y) - y/x}{\log x + x/y}$$

12. State the conditions for maxima and minima of $f(x,y)$

Soln:- [April/May - 2015]

If $f_x(a,b) = 0$, $f_y(a,b) = 0$ and $f_{xx}(a,b) = A$, $f_{xy}(a,b) = B$, $f_{yy}(a,b) = C$, then

(i) $f(a,b)$ is maximum value if $AC - B^2 > 0$ and $(A < 0)$ (or) $(B < 0)$

(ii) $f(a,b)$ is minimum value if $AC - B^2 > 0$ and $(A > 0)$ (or) $(B > 0)$

(iii) $f(a,b)$ is not an extremum value if $AC - B^2 < 0$

(iv) If $AC - B^2 = 0$, then test is inconclusive.

13. If $u = y/z + z/x + x/y$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Soln [Nov/Dec - 2014]

Given $u(x,y,z) = y/z + z/x + x/y$.

$$\Rightarrow u(tx, ty, tz) = \frac{ty}{tz} + \frac{tz}{tx} + \frac{tx}{ty} = t^0\left(\frac{y}{z} + z/x + x/y\right)$$

$\Rightarrow$ u is a homogeneous function of degree '0'

$\therefore$ By Euler's thm, $x\frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0 \cdot u = 0$

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14. Evaluate $\lim_{\substack{x \to \infty \\ y \to 2}} \left(\frac{xy+5}{x^2+2y^2} \right)$ (May / June – 2014)

Soln

$$\lim_{\substack{x \to \infty \\ y \to 2}} \left(\frac{xy+5}{x^2+2y^2} \right) = \lim_{x \to \infty} \left(\lim_{y \to 2} \frac{xy+5}{x^2+2y^2} \right)$$

$$= \lim_{x \to \infty} \left[\frac{2x+5}{x^2+8} \right]$$

$$= \lim_{x \to \infty} \left[\frac{x \cdot (2 + 5/x)}{x \, (x + 8/x)} \right]$$

$$= \lim_{x \to \infty} \left(\frac{2 + 5/x}{x + 8/x} \right)$$

$$= \frac{2 + 5/\infty}{\infty + 8/\infty} = \frac{2+0}{\infty + 0} = 2/\infty = 0.$$

15. If $x^y + y^x = c$ find dy/dx. [Jan – 2014]

Soln

Let $f(x,y) = x^y + y^x - c = 0.$

$$\frac{dy}{dx} = \frac{-(\partial f/\partial x)}{\partial f/\partial y} = \frac{-y x^{y-1} + y^x \log y}{x^y \log x + x y^{x-1}}$$

UNIT - IV

INTEGRAL CALCULUS

1. What is wrong with the eqn $\int_{-1}^{2} \frac{4}{x^3}\, dx = \left[\frac{-2}{x^2}\right]_{-1}^{2} = \frac{3}{2}$?

[Jan - 2018]

Soln:-

The integrand function $\frac{4}{x^3}$ is not continuous on $[-1, 2]$. Also $4/x^3$ has infinite discontinuity at $x=0$.

$\therefore \int_{-1}^{2} 4/x^3\, dx$ does not exist.

2. Evaluate $\int_{4}^{\infty} \frac{1}{\sqrt{x}}\, dx$ and determine whether it is convergent or divergent ?

[Jan - 2018]

Soln

$$\int_{4}^{\infty} \frac{1}{\sqrt{x}}\, dx = \lim_{t\to\infty} \int_{4}^{t} \frac{1}{\sqrt{x}}\, dx = \lim_{t\to\infty} \left(2\sqrt{x}\right)_{x=4}^{x=t}$$

$$= \lim_{t\to\infty} \left[2\sqrt{t} - 4\right] = 2(\infty) - 4 = \infty \text{ (does not exist)}$$

$\therefore$ The given integral is divergent

3. Evaluate $\int_{0}^{\pi/2} \log(\tan x)\, .dx$

Soln:-

Let $I = \int_{0}^{\pi/2} \log(\tan x).dx. \longrightarrow (1)$

Property: $\int_{0}^{a} f(x).dx = \int_{0}^{a} f(a-x)\, dx$

$$\therefore I = \int_{0}^{\pi/2} \log\left(\tan(\pi/2 - x)\right) dx = \int_{0}^{\pi/2} \log(\cot x).dx \longrightarrow (2)$$

$$(1) + (2) \Rightarrow 2I = \int_{0}^{\pi/2} \left[\log(\tan x) + \log(\cot x)\right] dx = \int_{0}^{\pi/2} \log(\tan x . \cot x)\, dx$$

$$2I = \int_{0}^{\pi/2} \log\left(\frac{\tan x}{\tan x}\right) dx = \int_{0}^{\pi/2} \log 1\, dx = 0 \quad (\because \log(1) = 0)$$

$\therefore I = 0$

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4. Evaluate $\int_{-1}^{1} \log\left(\frac{4-x}{4+x}\right) dx$

Soln

Property : $\int_{-a}^{a} f(x)\, dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd} \\ 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \end{cases}$

$f(x) = \log\left(\frac{4-x}{4+x}\right)$, $f(-x) = \log\left(\frac{4+x}{4-x}\right) = \log\left(\frac{4-x}{4+x}\right)^{-1}$

$= -\log\left(\frac{4-x}{4+x}\right)$

$= -f(x)$

$\therefore$ $f(x)$ is odd function

$\Rightarrow \int_{-1}^{1} \log\left(\frac{4-x}{4+x}\right) dx = 0.$

5. Evaluate $\int \frac{\sec^2(\log x)}{x}\, dx$

Soln

Let $I = \int \frac{\sec^2(\log x)}{x}\, dx.$

Sub : $u = \log x$, $du = \cancel{\log}\frac{1}{x}.dx$

$\therefore I = \int \sec^2 u \,.\, du = \int d(\tan u) = \tan u + C$

$= \tan(\log x) + C$

6. Evaluate $\int (\log x)^2 dx.$

Soln :-

Formula $\boxed{\int u\,dv = uv - \int v\,du}$

Let $u = (\log x)^2 \Rightarrow du = 2\log x \cdot \frac{1}{x}\, dx$

$dv = dx \Rightarrow v = x$

$\therefore \int (\log x)^2 dx = x(\log x)^2 - \int x \cdot 2\log x \cdot \frac{1}{x}\, dx$

$= x(\log x)^2 - 2\int \log x \cdot dx.$

$= x(\log x)^2 - 2\left[x.\log x - \int x \cdot \frac{1}{x}\, dx\right]$

$= x(\log x)^2 - 2\left[x\log x - x\right] + C.$

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7. Evaluate $\int_0^{\pi/2} \sin^7 x \, . dx$.

Soln

$$\int_0^{\pi/2} \sin^7 x \, dx = \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{16}{35}$$

8. Evaluate $\int_0^{\pi/2} \cos^6 x \, dx$

Soln :-

$$\int_0^{\pi/2} \cos^6 x \, dx = \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{15\pi}{96} = \frac{5\pi}{32}$$

9. Evaluate $\int \frac{1}{x^2 - a^2} dx$ using partial fraction.

Soln :-

$$\frac{1}{x^2-a^2} = \frac{1}{(x+a)(x-a)} = \frac{A}{(x+a)} + \frac{B}{(x-a)} = \frac{A(x-a)+B(x+a)}{x^2-a^2}$$

$$\therefore \quad 1 = A(x-a) + B(x+a) \longrightarrow (1)$$

Put $x = a$ in (1) $\Rightarrow$ $1 = B(2a)$ $\Rightarrow$ $\boxed{B = 1/2a}$

Put $x = -a$ in (1) $\Rightarrow$ $1 = A(-2a)$ $\Rightarrow$ $\boxed{A = -1/2a}$

$$\therefore \quad \frac{1}{x^2-a^2} = \frac{-1}{2a(x+a)} + \frac{1}{2a(x-a)}$$

$$\therefore \int \frac{1}{(x^2-a^2)} dx = -\frac{1}{2a}\int \frac{1}{(x+a)} dx + \frac{1}{2a}\int \frac{1}{(x-a)} dx$$

$$= -\frac{1}{2a} \log(x+a) + 1/2a \log(x-a)$$

$$= \frac{1}{2a}\left[\log\left\{\frac{x-a}{x+a}\right\}\right] + C$$

10. What is the difference between Proper and Improper integrals. Explain it by suitable example.

Soln :-

In a proper integral $\int_a^b f(x). dx$, it is assumed that the limits of integration are finite and that the integrand $f(x)$ is continuous for every value of x in the interval $a \le x \le b$. If atleast one of these conditions is violated (not satisfied), then the integral is known as improper integral.

Example:-

$$\int_1^2 x\, dx \quad \text{(Proper integral)}$$

$$\int_0^\infty \frac{1}{x}\, dx \quad \text{(Improper integral)}$$

11) State Bernoulli Formula and using it evaluate $\int x^2 \cos 5x \,.\, dx$.

Soln:-

Bernoulli Formula : $\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \dots$

where $u', u'', u''', \dots$ are successive differentiations of u

and $v_1, v_2, v_3, \dots$ are successive integrals of v.

Given $\int x^2 \cos 5x \,.\, dx$.

Take $u = x^2$ | $dv = \cos 5x$

$u = x^2$ | $v = \int dv = \int \cos 5x = \frac{\sin 5x}{5}$

$u' = 2x$ | $v_1 = -\frac{\cos 5x}{25}$

$u'' = 2$ | $v_2 = -\frac{\sin 5x}{125}$

$u''' = 0$ | ($\because u''' = 0$, we need not to find v_3)

$\therefore$ Using Bernoulli formula

$$\int x^2 \cos 5x \,.\, dx = \frac{x^2 \sin 5x}{5} - 2x\left(\frac{-\cos 5x}{25}\right) + 2\left(\frac{-\sin 5x}{125}\right)$$

$$= \frac{x^2 \sin 5x}{5} + \frac{2x \,.\, \cos 5x}{25} - \frac{2\sin 5x}{125} + C.$$

12) Fundamental theorem of calculus: If 'f' is a continuous function on $[a,b]$, then $\int_a^b f(x)\, dx = F(b) - F(a)$ where $F' = f$ [N/D-2018]

(13) Find the derivative of $G(x) = \int_x^1 \cos\sqrt{t}\, dt$ [N/D – 2019]

Soln

Given $G(x) = \int_x^1 \cos\sqrt{t}\, dt = -\int_1^x \cos\sqrt{t}\, dt$

Here $f(t) = \cos\sqrt{t}$ is continuous

$\therefore G'(x) = -\cos\sqrt{x}$.

---

(14) Determine whether the given integral $\int_0^\infty e^x\, dx$ is convergent or divergent [N/D – 2019]

Soln

$$\int_0^\infty e^x dx = \lim_{t\to\infty} \int_0^t e^x dx = \lim_{t\to\infty} \left[\{e^x\}_{x=0}^{x=t} \right]$$

$$= \lim_{t\to\infty} \left[e^t - e^0 \right]$$

$$= e^\infty - 1$$

$$= \infty - 1$$

$$= \infty$$

$\therefore$ Given integral is devergent.

---

(15) Evaluate $\int_0^{\pi/2} \frac{1}{1+\tan x} dx$. [A/M – 2019]

Soln

Let $I = \int_0^{\pi/2} \frac{1}{1+\tan x} dx = \int_0^{\pi/2} \frac{1}{1+\left(\frac{\sin x}{\cos x}\right)} dx = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx \rightarrow (1)$

Take $f(x) = \frac{\cos x}{\cos x + \sin x}$

$$\therefore f\left(\pi/2 - x\right) = \frac{\cos\left(\pi/2 - x\right)}{\cos\left(\pi/2 - x\right) + \sin\left(\pi/2 - x\right)} = \frac{\sin x}{\sin x + \cos x}.$$

Using $\boxed{\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx}$ → property

$$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx \longrightarrow (2)$$

(1) + (2) $\Rightarrow 2I = \int_0^{\pi/2} dx = \pi/2$.

$\Rightarrow \boxed{I = \pi/4}$

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(16) Evaluate $\int_3^\infty \frac{dx}{(x-2)^{3/2}}$ and determine whether it is convergent or divergent. [A/M – 2019]

Soln

$$\int_3^\infty \frac{dx}{(x-2)^{3/2}} = \lim_{t\to\infty} \int_3^t \frac{dx}{(x-2)^{3/2}} = \lim_{t\to\infty} \int_3^t (x-2)^{-3/2}\, dx$$

$$= \lim_{t\to\infty} \left[\frac{(x-2)^{-3/2+1}}{(-3/2+1)} \right]_3^t = \lim_{t\to\infty} \left[\frac{(x-2)^{-1/2}}{-1/2} \right]_3^t = -2 \lim_{t\to\infty} \left(\frac{1}{\sqrt{x-2}} \right)_3^t$$

$$= -2 \lim_{t\to\infty} \left(\frac{1}{\sqrt{t-2}} - \frac{1}{\sqrt{3-2}} \right) = -2 \left(\frac{1}{\sqrt{\infty-2}} - \frac{1}{1} \right)$$

$$= -2 \left(\frac{1}{\infty} - 1 \right)$$

$$= -2(0-1)$$

$$= 2.$$

---

(17) If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$. [N/D – 2018]

Soln

Take $2x = t$ | When $x=0 \Rightarrow t=0$

$\Rightarrow 2\,dx = dt$ | $x=2 \Rightarrow t = 2(2) = 4.$

$$\therefore \int_0^2 f(2x)\,dx = \int_0^4 f(t)\cdot\frac{dt}{2} = \frac{1}{2}\int_0^4 f(x)\cdot dx = \frac{1}{2}(10) = 5.$$

---

(18) Determine whether integral $\int_1^\infty \frac{\ln(x)}{x}\,dx$ is convergent or divergent. Evaluate if it convergent. [Nov/Dec – 2020]

Soln

$$\int_1^\infty \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln(x)}{x}\,dx. \longrightarrow (1)$$

Sub: $u = \ln(x)$ | When $x = t \Rightarrow u = \ln(t)$

$\therefore du = \frac{1}{x}\,dx$ | $x = 1 \Rightarrow u = \ln(1) = 0.$

(1) becomes

$$\therefore \int_1^\infty \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty} \int_0^{\ln(t)} u\,du = \lim_{t\to\infty} \left[\frac{u^2}{2} \right]_0^{\ln(t)} = \frac{1}{2}(\ln\infty - 0)$$

$$= \frac{1}{2}(\infty - 0)$$

$$= \infty.$$

$\therefore$ Given integral is divergent.

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1) Evaluate $\int_{-1}^{2}\int_{x}^{x+2} dy\, dx$. [April/May – 2018], [Jan – 2018]

Soln

$$\int_{-1}^{2}\left[\int_{x}^{x+2} dy\right]dx = \int_{-1}^{2}\left[y\right]_{y=x}^{y=x+2} dx = \int_{-1}^{2}(x+2-x)\,dx$$

$$= 2\int_{-1}^{2} dx = 2\left[x\right]_{x=-1}^{x=2} = 2(2+1) = 6$$

2) Evaluate $\int_{0}^{2\pi}\int_{0}^{\pi/2}\int_{0}^{4} r^3 \sin\theta\, dr\, d\theta\, d\phi$. [April/May – 2018] [Jan – 2018]

Soln :-

$$\int_{0}^{2\pi}\int_{0}^{\pi/2}\left(\int_{0}^{4} r^3 \sin\theta\, dr\right) d\theta\, d\phi = \int_{0}^{2\pi}\int_{0}^{\pi/2}\left(\frac{r^4}{4}\right)_{r=0}^{r=4} \sin\theta\, d\theta\, d\phi$$

$$= 4^3 \int_{0}^{2\pi}\left(\int_{0}^{\pi/2} \sin\theta\, d\theta\right) d\phi$$

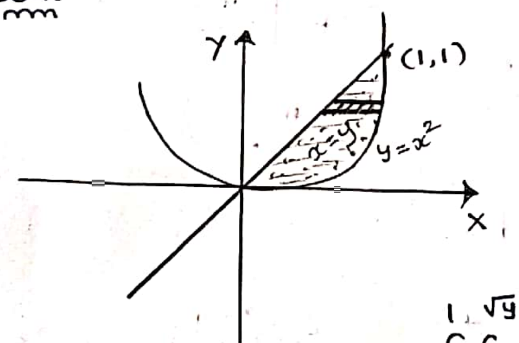
$$= 4^3 \int_{0}^{2\pi}\left(-\cos\theta\right)_{\theta=0}^{\theta=\pi/2} d\phi = 4^3 \int_{0}^{2\pi} 1.\, d\phi$$

$$= 64\left[\phi\right]_{\phi=0}^{\phi=2\pi} = 64(2\pi)$$

$$= 128\pi$$

3) Find the area bounded by the line $y = x$ and parabola $x^2 = y$ [Nov/Dec – 2017]

Soln



	low	upp
x	y	$\sqrt{y}$
y	0	1

$$\text{Required Area} = \int_{0}^{1}\int_{y}^{\sqrt{y}} dx\, dy = \int_{0}^{1}\left(y^{1/2} - y\right) dy$$

$$= \left(\frac{y^{3/2}}{3/2} - \frac{y^2}{2}\right)_{y=0}^{y=1} = \frac{1}{3/2} - \frac{1}{2}$$

$$= \frac{2}{3} - \frac{1}{2}$$

$$= \frac{4-3}{6} = \frac{1}{6} \text{ Square unit}$$

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4) Evaluate $\int_1^3\int_2^3\int_1^2 x^2yz\,dx\,dy\,dz$. [Nov/Dec - 2017]

Soln

$$\int_1^3\int_2^3\int_1^2 x^2\,yz\,dx\,dy\,dz = \int_1^3\int_2^3 yz.\left[\frac{x^3}{3}\right]_{x=1}^{x=2} dy\,dz$$

$$= \int_1^3\int_2^3 yz\left(\frac{8}{3}-\frac{1}{3}\right) dy\,dz = \frac{7}{3}\int_1^3\left(\int_2^3 yz\,dy\right)dz$$

$$= \frac{7}{3}\int_1^3\left[\frac{zy^2}{2}\right]_{y=2}^{y=3} dz = \frac{7}{3}\int_1^3 z\left(\frac{9}{2}-\frac{4}{2}\right)dz$$

$$= \frac{7}{3}\times\frac{5}{2}\int_1^3 z\,dz = \frac{35}{6}\left[\frac{z^2}{2}\right]_{z=1}^{z=3}$$

$$= \frac{35}{12}\left[9-1\right] = \frac{35\times 8}{12} = \frac{70}{3}$$

5) Evaluate $\int_0^{\pi}\int_0^{\sin\theta} r\,dr\,d\theta$. [Apr/May - 2017] [May 2014]

Soln

$$\int_0^{\pi}\left(\int_0^{\sin\theta} r\,dr\right)d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_{r=0}^{r=\sin\theta} d\theta = \frac{1}{2}\int_0^{\pi}\sin^2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \frac{1}{2\times 2}\int_0^{\pi}(1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{4}\left[\theta-\frac{\sin 2\theta}{2}\right]_{\theta=0}^{\theta=\pi} = \frac{1}{4}\left[\{\pi-0\}-\{0-0\}\right]$$

$$= \frac{\pi}{4}$$

6) Evaluate $\int_1^3\int_3^4\int_1^4 xyz\,dx\,dy\,dz$. [Apr/May - 2017]

Soln

$$\int_1^3\int_3^4\left(\int_1^4 xyz\,dx\right)dy\,dz = \int_1^3\int_3^4 yz\left(\frac{x^2}{2}\right)_{x=1}^{x=4} dy\,dz$$

$$= \frac{15}{2}\int_1^3\left(\int_3^4 yz\,dy\right)dz = \frac{15}{2}\int_1^3 z\left(\frac{y^2}{2}\right)_{y=3}^{y=4} dz$$

$$= \frac{15}{2}\times\frac{7}{2}\int_1^3 z\,dz = \frac{15\times 7}{4}\left[\frac{z^2}{2}\right]_{z=1}^{z=3} = \frac{105}{8}\left[9-1\right] = 105$$

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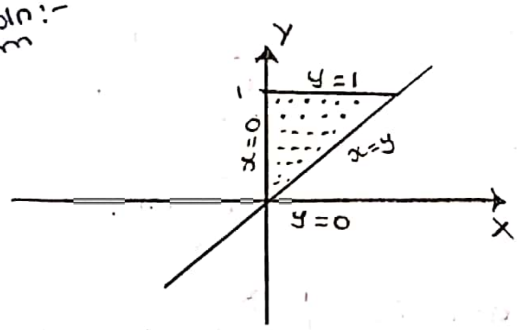
7. Change the order of integration in $\int_0^1 \int_0^y f(x,y)\, dx\, dy$. ③

[NOV/DEC - 2016]

Soln:-



After changing

	low	upp
x	0	1
y	x	1

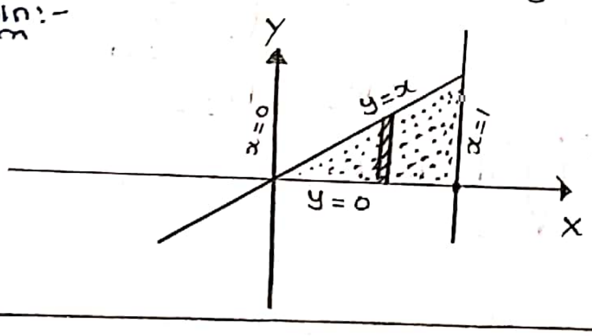
∴ By changing the order of integration, given integration is equal to $\int_0^1 \int_x^1 f(x,y)\, dy\, dx$.

8. Evaluate $\int_0^1 \int_0^2 \int_0^3 xyz\, dx\, dy\, dz$ [NOV/DEC - 2016]

(like prob: 6) Try Yourself. (Ans: $\frac{36}{8}$)

9. Sketch the region of integration in $\int_0^1 \int_0^x dy\, dx$ [May/June - 2016]

Soln:-

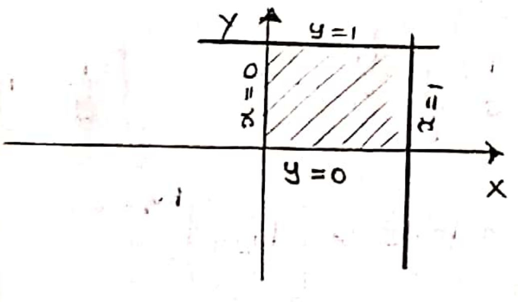


	low	upp
x	0	1
y	0	x

10. Find the area bounded by the lines $x=0$, $y=1$, $x=1$, & $y=0$.

Soln:-

[May/June - 2016]



$$\text{Required area} = \int_0^1 \int_0^1 dx\, dy$$

$$= \int_0^1 [x]_{x=0}^{x=1}\, dy$$

$$= 1 \int_0^1 dy = 1\, [y]_{y=0}^{y=1}$$

$$= 1 \text{ sq. unit}$$

11. Evaluate $\int_0^{\pi}\int_0^{a} r\,dr\,d\theta$. [Dec – 2015], [Jan – 2014]

Soln
$$\int_0^{\pi}\left(\int_0^{a} r\,dr\right)d\theta = \int_0^{\pi}\left(\frac{r^2}{2}\right)_{r=0}^{r=a} d\theta = a^2/2 \int_0^{\pi} d\theta$$
$$= a^2/2 \cdot \pi .$$

12. Sketch the region of integration in $\int_0^1\int_0^x dy\,.dx$ [Dec – 2015]

(Refer prob: 9)

13. Evaluate $\int_1^2\int_1^3 \frac{dx\,dy}{x\,y}$. [Apr / May – 2015]

Soln:-
$$\int_1^2\left(\int_1^3 \frac{dx}{xy}\right)dy = \int_1^2 \frac{1}{y}\left(\int_1^3 \frac{dx}{x}\right)dy$$
$$= \int_1^2 1/y \left[\log x\right]_{x=1}^{x=3} dy = (\log 3 - \log 1)\int_1^2 \frac{dy}{y}$$
$$= \log 3 \left(\log y\right)_{y=1}^{y=2} \quad (\because \log 1 = 0)$$
$$= \log 3 \times \log 2$$

14. Obtain the value of $\int_0^a\int_0^b\int_0^c dx\,dy\,dz$. [Apr / May – 2015]

Soln

Try Yourself

(like prob: 4) (Ans: abc)

15. Evaluate $\int_2^3\int_1^2 \frac{1}{xy}\,dx\,dy$. [Nov / Dec – 2014]

Soln:-
$$\int_2^3\left(\int_1^2 \frac{1}{xy}\,dx\right)dy = \int_2^3 \frac{1}{y}\left(\int_1^2 \frac{dx}{x}\right)dy = \int_2^3 \frac{1}{y}\left(\log x\right)_{x=1}^{x=2} dy$$
$$= \log 2 \int_2^3 \frac{dy}{y} = \log 2 \times \left(\log y\right)_{y=2}^{y=3}$$
$$= \log 2 \left[\log 3 - \log 2\right].$$

16. Evaluate $\int_0^{\pi/2}\int_0^{\sin\theta} r\,d\theta\,.dr$. [Nov / Dec – 2014]

Soln

Correct form is $\int_0^{\pi/2}\int_0^{\sin\theta} r\,dr\,d\theta$. Ans: $\pi/4$

(Refer prob: 5)

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17. Evaluate $\int_0^5 \int_0^2 (x^2+y^2)\,dx\,dy$ (May/June - 2014) (5)

Soln :-

$$\int_0^5 \left(\int_0^2 (x^2+y^2)\,dx \right) dy = \int_0^5 \left(\frac{x^3}{3} + y^2 x \right)_{x=0}^{x=2} dy$$

$$= \int_0^5 \left(8/3 + 2y^2 \right) dy = \left(8/3\, y + 2y^3/3 \right)_{y=0}^{y=5}$$

$$= \left(40/3 + \frac{250}{3} \right) = \frac{290}{3}$$

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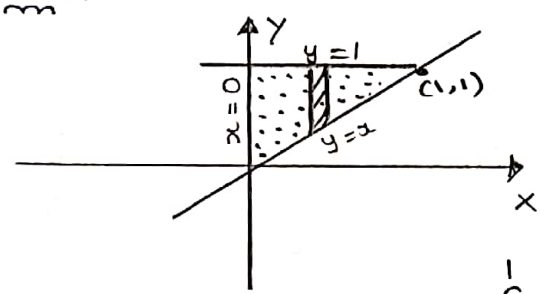
18. Find the area bounded by the lines $x=0$, $y=1$ & $y=x$ using double integration. [Jan - 2014]

Soln :-



	low	upp
x	0	1
y	x	1

Required area $= \int_0^1 \int_x^1 dy\,dx = \int_0^1 (1-x)\,dx$

$$= \int \left(x - x^2/2 \right)_{x=0}^{x=1} = 1 - 1/2 = 1/2 \text{ sq.unit}$$

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